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LETTER TO THE EDITOR

Hull percolation

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Abstract. We present a novel problem of random tiling of the plane with two kinds of squares having two quarters of circles drawn on them. The lines generated by these pieces of a circle are shown to form fractal loops, the nature of which is equivalent to 'smart kinetic walks', to hulls of percolation clusters or, finally, to conformations of polymers at the θ point. We also study the transport properties of these structures and of the 'modified' percolation problem associated with it.

Let us consider the tiling of a plane with two elements, shown in figures 1(a) and (b), chosen at random with equal probability. Both cells consist of two quarters of circle, whose centres are located on two opposite corners of the square, and whose radii are equal to half the length of the square. By construction, these pieces of a circle form continuous lines. It is thus impossible to have branching points or to find dead ends. Therefore these lines must either be closed (and form loops) or be infinite (i.e. start and end at 'infinity').

This letter is devoted to some properties of this tiling and its relation to various models of statistical mechanics. We call this problem 'hull percolation' for reasons that will become clear later.

(a) *Loop distribution.* We generated one large lattice (1000×1000) and recorded the number, $N(L)$, of closed loops of given size L (the length of the loop is counted

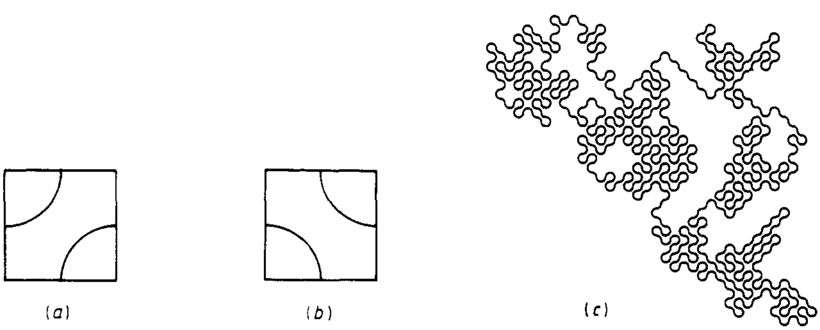

Figure 1. We consider in this letter the random tiling of a plane with the two elements shown in (a) and (b). The quarters of circles are found to generate loops as shown in (c) with a fractal structure.

as the number of quarters of a circle). We found that this distribution, shown in figure 2, follows a power law:

$$N(L) \propto L^{-\theta} \tag{1}$$

where $\theta = 2.16 \pm 0.10$.

(*b*) *Fractal dimension.* Using the construction of this tiling, it is straightforward to generate a single loop without generating a complete tiling over the plane. It is sufficient to decide for the orientation of the squares only along the line being constructed and to test at every step of the generation whether the new square reached by the line had not been encountered previously. In this case, we use the piece of circle already chosen that may either continue the line, and avoid a crossing, or close the loop. This algorithm is thus very close in spirit to the one introduced by Ziff *et al* [1], and developed by Ziff and Sapoval [2], to generate external perimeters of percolation clusters. We will see, in the following, that this analogy is not fortuitous. We used periodic boundary conditions on a $L \times L$ square lattice, which turned it into a torus. We measured the average length of loops drawn on this torus if and only if they had a genus different from $(0, 0)$. The genus of the loop, (a, b), represents the number of times (winding number) the loop crosses the horizontal boundary a and the vertical boundary b. This topological invariant counts the number of windings made on the torus. We only use it here as an easy criterion for deciding if the loop spans the lattice or not. The average, $\langle M \rangle$, of the length of these loops depends on the lattice size L, according to a power-law relation:

$$\langle M \rangle \propto L^{d_f} \tag{2}$$

where d_f is the fractal dimension of the loops. Using data obtained on 500 non-zero genus loops on lattices of size 10, 20, 30, 40 and 50, we obtained the estimate $d_f = 1.73 \pm 0.02$, as shown in figure 3.

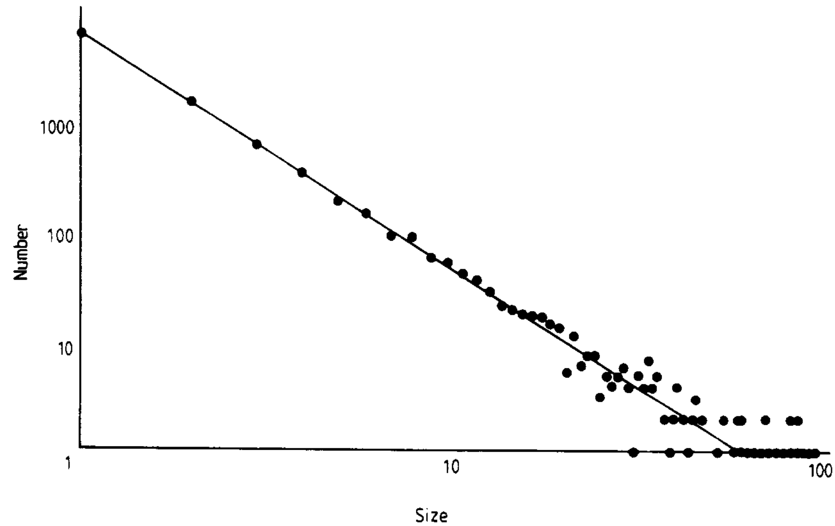

Figure 2. Loop size distribution obtained in a 1000×1000 lattice. The total number of loops amounts to 100 000. The graph shows a log-log plot of the histogram showing a power-law distribution (see (1) in the text).

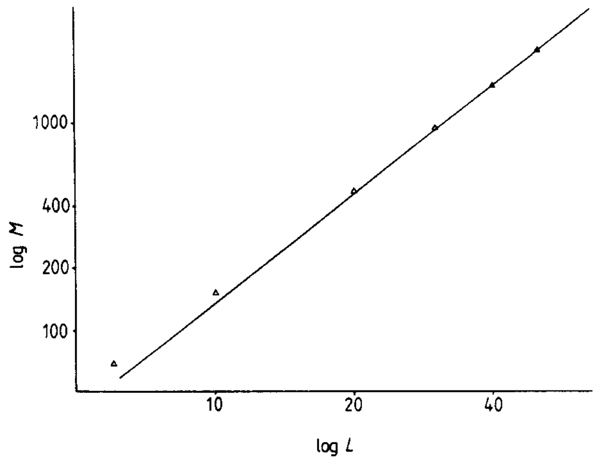

Figure 3. Mean length of a spanning loop generated in a torus of size $L \times L$. The spanning criterion is that the loops have non-zero genus. The log-log plot of this mean length against size L gives the fractal dimension as the slope of the line drawn in the figure. We found here that 1.73 ± 0.02.

(*c*) *Probability of spanning.* We also studied the fractal dimension using a different method. We considered a long strip of width w and of length 10^4. We computed the probability for a line to connect the two longitudinal borders of the strip, $P(w)$ (we will call this the 'probability of spanning' in the following). The use of such a strip geometry is usual in transfer matrix methods [3] so as to obtain a good average of the property considered independent of the length. The above-mentioned length was sufficient to ensure small statistical fluctuations. We also recorded the length M' of all the spanning lines. The evolution of $\langle M' \rangle$ scales with the strip width as (figure 4)

$$\langle M' \rangle \propto w^{d_f} \tag{3}$$

where $d_f = 1.76 \pm 0.04$. These two determinations of the fractal dimension are consistent.

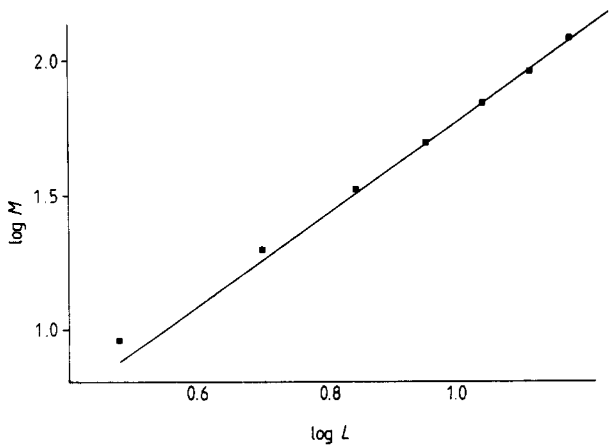

Figure 4. Mean length of a spanning line in a strip-shape geometry plotted against strip width in a log-log scale. The slope should give also the fractal dimension of the lines generated in the tiling. We found 1.76 ± 0.04 in agreement with the previous determination (figure 3).

The probability of spanning $P(w)$ decreases as the width increases with a power law which is

$$P(w) \propto w^{-\alpha}. \tag{4}$$

Figure 5 shows a log-log plot of $P(w)$ against w, which provides the estimate of $\alpha = 0.95 \pm 0.05$. In fact, assuming the property of isotropy of these lines we are led to expect spanning lines through the strip separated by a distance roughly equal to the width of the strip. Thus $\alpha = 1$, consistent with our numerical estimate.

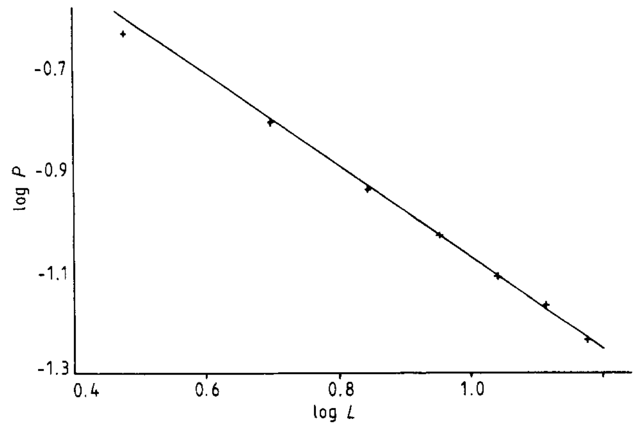

Figure 5. A log-log plot of the probability that a line crosses the strip as a function of the strip width. We found a power law of slope 0.95 ± 0.05 (see (4)).

(*d*) *Conductivity and elasticity.* The last scaling laws, equations (3) and (4), allow one to obtain information about the conductivity of the strip when one considers that the quarters of circles have a given resistance. Because there is no branching point, nor dead end, the conductance of a spanning line is inversely proportional to its length. Therefore the conductivity $G(w)$ of the strip is equal to the probability $P(w)$ times the inverse of $\langle M' \rangle$ if these two properties are decorrelated. Finally

$$G(w) = \langle (P(w)/M'(w)) \rangle = P(w)/\langle M'(w) \rangle \tag{5}$$

and we obtain

$$G(w) \propto w^{-g} \tag{6}$$

where $g = \alpha + d_f = 2.70 \pm 0.07$ (or $g = 2.75$ using the values $\alpha = 1$ and $d_f = 1.75$). Similar forms of the power-law dependence of conductivity plotted against system size are known to occur, for instance, in percolation. However, the value of the exponent g obtained here is remarkably large (for percolation the corresponding exponent amounts to $t/\nu = 0.97 \pm 0.01$ [3] where t denotes the conductivity exponent and ν that of the correlation length).

If we consider now the case where the pieces of circles are elastic wires, then the elastic modulus of the structure is clearly controlled by bending elasticity. Therefore, taking into account lever-arm effects and assuming that the conformations of the spanning lines are statistically isotropic (or at least that the radius of gyration of these lines scales in the same way in all directions, parallel or orthogonal to the axis of the strip), we can obtain the evolution of the elastic modulus, $E(w)$, with the system size following the argument given in [4]

$$E(w) \propto w^{-\varepsilon} \qquad (7)$$

where $\varepsilon = g+2$, thus giving $\varepsilon = 4.70 \pm 0.07$, still much larger than the corresponding value for percolation: $\tau/\nu = 2.97 \pm 0.03$ [5] (where τ is the elasticity exponent for percolation for systems having angular elasticity). Let us note, however, that the relation between ε and g is exact in our case whereas it is only an upper bound for percolation $\tau/\nu \leq t/\nu + 2$ [4].

We now look at the connection with other models.

(*a*) *The smart kinetic walk* (SKW). Among the different models for self-avoiding walks or polymers (self-avoiding walk, true self-avoiding walk, kinetic growing walk, Laplacian walk, etc) (for a review see [6]) one model has been considered, the 'smart kinetic walk' [7], which shares a lot of properties with the loops we have presented above. In addition to self-avoidance, the SKW has the 'smartness' of not getting trapped. Such a walk cannnot get trapped into a loop made by its past trajectory that would impel its further growth. In two dimensions, such smartness can be implemented by only *local* information: whenever a walker gets to the neighbourhood of its past trajectory, one needs to know the direction in which the trajectory was followed. The walker at this point should turn in the opposite direction.

In our model of tiling, we always set *two* quarters of a circle per square. A line will go through one of these arcs, while the other arc is left so as to provide the local information needed to get self-avoidance of these walks. Moreover, in order to avoid trapping, one also needs to know the direction followed by the walk. Let us note that, whenever a walk has started (for example see figure 6) in A, with a given direction indicated by an arrow, then the direction followed by the walker on all the available edges of the plane is fixed once and for all. These directions are indicated by arrows on figure 6. Therefore, knowing the location of the walker at a given time is enough to know its direction. This simple property prevents the walker from being trapped. The conclusion is that this walk is in the same universality class as SKW.

Our numerical results strengthen this conclusion. The fractal dimension of SKW has been shown indirectly [7, 8] to be exactly equal to $\frac{7}{4} = 1.75$, in agreement with our numerical estimates.

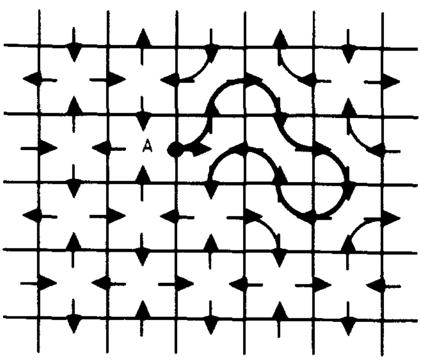

Figure 6. This figure shows that each edge of squares in the plane can be reached with only one direction once the walk has started in A. Therefore, the structure of the tiling provides not only self-avoidance but also 'smartness' in preventing the walker from being trapped.

(*b*) *The hull of percolation clusters.* In [7], it was proven that SKW belongs to the same universality class as the hull of a percolation cluster. The hull is defined as the set of vacant bond nearest neighbours to a cluster that can be joined by a continuous path from infinity without crossing the cluster. In other words, it is the external perimeter. This suggests we can translate some known results of percolation to our problem. Provided that not only the structure of individual loops is identical in both problems, but also the complete statistical distribution of their sizes, we can obtain an estimate of the exponent θ introduced previously. At percolation threshold, we know that the distribution of clusters follows a power law given by

$$n(s) \propto s^{-\tau} \tag{8}$$

where $\tau = \frac{187}{91}$. In addition, it is known (see [8, 9]) that the relation between the length of the external hull, h, of a cluster and its area, s, is

$$h \propto s^{x} \tag{9}$$

where $x = \frac{12}{13}$. The combination of these two laws gives

$$N(h)\,\mathrm{d}h \propto n(s)\,\mathrm{d}s \tag{10}$$

$$\theta = 1 + (\tau - 1)/x \tag{11}$$

or numerically, $\theta = 2.143\ldots$, in perfect agreement with our estimate.

Let us call Λ the radius of gyration of a loop of length L. The distribution, $P(\Lambda)$, of Λ can be expressed in terms of $N(L)$, since $L \propto \Lambda^{d_f}$

$$P(\Lambda) = N(L)\mathrm{d}L/\mathrm{d}\Lambda \propto L^{-\theta}\Lambda^{d_f - 1} \propto \Lambda^{d_f(1-\theta)-1}. \tag{12}$$

The number of loops whose radius is larger than w is

$$\int_w^\infty N(\Lambda)\,\mathrm{d}\Lambda \propto w^{d_f(1-\theta)}. \tag{13}$$

In order to compute the probability that such a loop crosses the strip of width w, one must take into account an additional factor of w, which represents the possibility of having the centre of the loop located anywhere inside the loop. Thus, finally, the probability of spanning will vary as $w^{d_f(1-\theta)+1}$ or $\alpha = d_f(\theta - 1) - 1$—using the values of d_f and θ give $\alpha = 1$. This result is in complete agreement with both our numerical estimate $\alpha = 0.95$ and the isotropy argument we gave above leading to $\alpha = 1$.

The analogy of our random tiling with a percolation hull problem, which has been obtained indirectly, suggests looking for a more direct analogy.

Let us consider the following 'non-standard' percolation problem: on a square lattice, one cuts *all* squares along *one, and only one*, diagonal, either from top left to bottom right with probability $\frac{1}{2}$, or from top right to bottom left, with the same probability (see figure 7). These cuts will form clusters, the hulls of which are the loops introduced in our random tiling problem. Indeed, in figure 1, we see in the tiling squares (*a*) and (*b*) both types of connection still preserved by the two possible diagonal cuts. Therefore, our lines just reflect the hull of these clusters. This problem does not seem to be a percolation problem strictly speaking, since we impose one fixed occupancy level instead of using it as a free parameter as would be done in normal percolation. However, in figure 7, we can see clearly that the bonds are drawn on two square lattices. One of them is obtained from the other one, by translating it by half the lattice spacing, in the two directions of the principal axis. If we now focus on only one

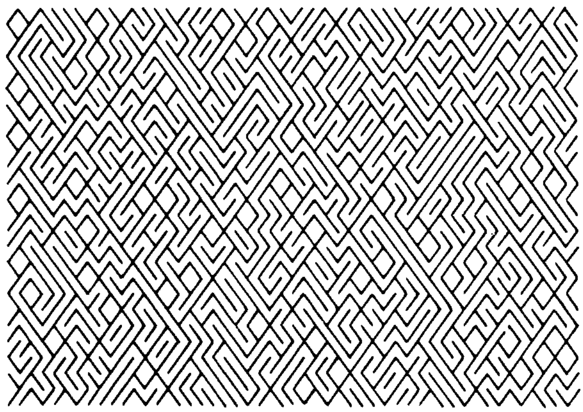

Figure 7. The tiling problem can be seen as the hull of a 'non-standard' percolation problem defined in the following way. Each square of a lattice is cut along one diagonal at random. These cuts form clusters, the hull of which are in direct correspondance with the loops drawn by our tiling model.

lattice, we see that the bonds will be present with a probability of 0.5. Therefore, for both of these lattices, we have precisely a bond percolation problem at threshold. Let us also note that the geometry of one lattice is exactly dual to the other. Therefore, in figure 7, we have two dual percolation configurations at p_c.

We indeed checked numerically that the scalar transport properties of this non-standard percolation is identical to the one observed in usual percolation. Explicitly, we studied by a transfer matrix analysis [3] the conductivity, G, of a strip variable width w and of length 20 000 assuming that the present diagonals were Ohmic resistors. We obtained the data shown in figure 8, in a log-log plot of G against w. The line drawn on this figure has the slope obtained for usual percolation, i.e. $t/\nu = 0.97$ [3]. It seems that we tend to recover such a value for large enough widths.

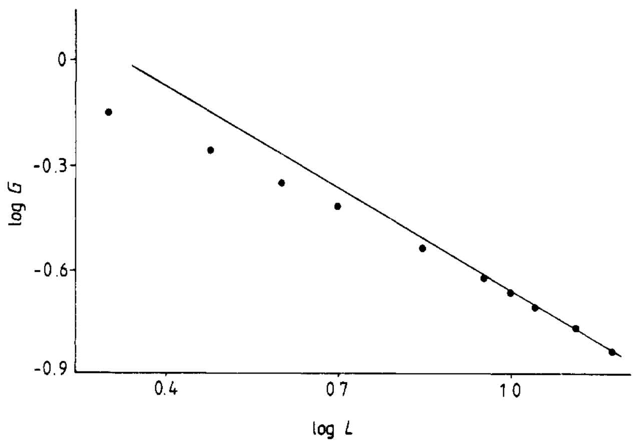

Figure 8. A log-log plot of the conductivity G of a strip of width w, whose geometry is identical to the one shown in figure 7 assuming that the bonds drawn are Ohmic resistors. The line drawn represents the slope expected for usual percolation. It seems likely that both the geometrical and transport features of normal percolation are recovered in this problem.

This provides a natural physical realisation of systems where the transport exponents computed previously in the case of electrical conductivity and elasticity should be observed.

A way to escape from criticality could be, for instance, to add a proportion of squares with either four quarters of circles drawn on them (the superposition of the squares shown in figures 1(a) and (b)) or none (blank squares).

We note that the hull of percolation clusters has been shown [10] to belong to the same universality class as polymers at the θ point [11]. Thus, our walks must also in turn belong to the same family.

We have considered the properties of a very simple model of random tiling of a plane which turned out to be equivalent to other models of statistical physics, as the problem of smart self-avoiding walks, hulls of percolation clusters and polymers at the θ point. These relations allowed the identification of a non-standard percolation problem underlying the notion of 'hulls' generated in the tiling. Our numerical results are consistent with known properties. In addition, we have obtained some new results concerning the transport properties of these structures, which can easily be tested experimentally, in the 'non-standard' percolation model introduced above.

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